THERMOELASTIC ANALYTICAL SOLUTION FOR 2D COMPOSITE LAMINATES

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Abstract

Thermo-mechanical phenomena are inherently related to composite materials, from manufacturing to service applications. Therefore, analytical and numerical models developed to simulate composite structural behaviour must satisfactorily account for thermal effects. Pagano's solutions for 2D and 3D composite problems are usually the base for comparison of the different theories and finite elements developments, even when thermo-mechanical behaviour is assessed. Although a number of papers in the literature use numerical results based on this solution, the formulation accounting for temperature effects is not explicitly presented, nor discussed. The objective of the present paper is to present Pagano's solution equations for a 2D case of a simply supported beam under a constant temperature field. Results obtained with the derived solution are discussed and compared against a 2D solid Finite Element Model (FEM) generated using a commercial software package.

1. INTRODUCTION

The development of laminate theories and elements to model composite structures has been the focus of many researches. The recent reviews by Sayyad and Ghugal [1], [2] together present more than 800 references on the subject. These extensive studies present an interesting concluding remark for future research: there is a need of studying problems involving thermo-mechanical loads in laminated and sandwich composite structures.

It is important to have reliable references to assess the capacities of newly developed models and theories, such as analytical solutions for 2D and 3D problems. Pagano's works for beams [3] and plates [4] appear as the most common base of comparison for laminated structures. Despite its spread use, thermal effects are not accounted for on the original formulation.

Recent works by Qian *et al*. present the analytical solutions for layered rectangular plates [5] and cylindrical arches [6] subjected to thermo-loads. These works solve the heat conduction equations for the layers, but a simpler approach is proposed in the present work.

The present paper presents the inclusion of a temperature field in Pagano's solution for 2D composite laminates under cylindrical bending [3]. The derived formulation is compared to finite element commercial solutions using 2D elements and the results for this apparently simple problem are discussed, revealing an important conclusion concerning the modelling of composite structures using solid elements.

2. FORMULATION

The constitutive equation for an orthotropic material considering thermal effects is:

$$\begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_y - \alpha_y \Delta T \\ \varepsilon_z - \alpha_z \Delta T \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \quad (1)$$

Where α_i is the expansion coefficient in *i*-direction. The coordinate systems and layer numbering scheme are kept the same [3]. Assuming plane stress, $\sigma_z = \tau_{xz} = \tau_{xz} = 0$, the constitutive relations then become:

$$\varepsilon_x - \alpha_x \Delta T = S_{11}\sigma_x + S_{12}\sigma_y = R_{11}\sigma_x + R_{12}\sigma_y \quad (2)$$

$$\varepsilon_y - \alpha_y \Delta T = S_{12}\sigma_x + S_{22}\sigma_y = R_{12}\sigma_x + R_{22}\sigma_y$$

$$\gamma_{xy} = S_{66}\tau_{xy} = R_{66}\tau_{xy}$$

Therefore the R_{ij} coefficients of Pagano's original work become equal to S_{ij} in Eq. (2). The original paper by Pagano assumes plane strain, therefore the R_{ij} coefficients values are different. The equations developed in the sequence are applicable also for plane strain, requiring only to change the definition of R_{ij}.

The equilibrium equations of the 2D problem are given as:

$$\sigma_{x,x} + \tau_{xy,y} = 0 \;\; ; \;\; \sigma_{y,y} + \tau_{xy,x} = 0 \quad (3)$$

The strain displacement relations are assumed according small displacement gradients hypothesis:

$$\varepsilon_x = u_{,x} \; ; \; \varepsilon_y = v_{,y} \; ; \; \gamma_{xy} = u_{,y} + v_{,x} \quad (4)$$

The boundary conditions are the same considered on Pagano's original work [3] for bending, a distributed load *q* on the upper surface and simply supported edges, according to Eq. (5).

$$\sigma_y\left(x, \frac{h}{2}\right) = q \; ; \; \sigma_y\left(x, -\frac{h}{2}\right) = \tau_{xy}\left(x, \pm\frac{h}{2}\right) = 0 \quad (5)$$

$$\sigma_y(0, y) = \sigma_y(l, y) = 0$$

$$v(0, y) = v(l, y) = 0$$

Where *q* has a sinusoidal distribution given by:

$$q\,(x) = q_0 \sin px \;\; ; \;\; p = p(n) = \frac{n\pi}{l} \quad (6)$$

This shape of the load function is suitable for the solution of the differential equations. If one wants to represent a particular load distribution, it can be expanded as a Fourier series and Pagano's solution is still applicable, as will be discussed in the next section.

In order to include the temperature field the same sinusoidal representation is needed as the same resources of the Fourier series apply for more general fields.

$$\Delta T\ (x) = \Delta T_0 \sin px \tag{7}$$

Although the temperature effects are included, an analogous solution for stresses to the one proposed in Pagano's original work [3] applies:

$$\begin{aligned} \sigma_x^{(i)} &= f_i''(y) \sin px \\ \sigma_y^{(i)} &= -p^2 f_i(y) \sin px \\ \tau_{xy}^{(i)} &= -p\ f_i'(y) \cos px \end{aligned} \tag{8}$$

Using Eqs. (2), (4) and (8), we find that the new functions $f_i(y)$ are defined by the solution of the following ordinary differential equation:

$$R_{11}^{(i)} f_i''''(y) - \left(R_{66}^{(i)} + 2R_{12}^{(i)}\right) p^2 f_i''(y) + R_{22}^{(i)} p^4 f_i(y) = \alpha_y^{(i)} \Delta T_0 p^2 \tag{9}$$

Where:

$$a_i = R_{66}^{(i)} + 2R_{12}^{(i)}\ ;\ b_i = \left(a_i^2 - 4R_{11}^{(i)} R_{22}^{(i)}\right)^{1/2}\ ;\ c_i = 2R_{11}^{(i)} \tag{10}$$

This is almost the same equation of Pagano's original work; moreover, the coefficients a_i, b_i and c_i remain as previously defined **[1]**. The addition of thermal effects introduces a particular solution for the differential equation, where the homogeneous solution is kept equal to the previously defined by Pagano.

$$f_i(y) = f_{ih}(y) + f_{ip}(y) \tag{11}$$

The particular solution is given by:

$$f_{ip}(y) = \frac{\alpha_y^{(i)} \Delta T_0}{R_{22}^{(i)} p^2} \tag{12}$$

There is a term dependent on α_y but not on α_x because the temperature distribution is assumed constant through *y*-direction, thus in the derivation of Eq. (9) the term related to α_x vanishes. Therefore the solution for the stresses become:

$$\begin{aligned} \sigma_x^{(i)} &= f_i''(y) \sin px = f_{ih}''(y) \sin px \\ \sigma_y^{(i)} &= -p^2 \left(f_{ih}(y) + \frac{\alpha_y^{(i)} \Delta T_0}{R_{22}^{(i)} p^2} \right) \sin px = -p^2 f_{ih}(y) \sin px - \frac{\alpha_y^{(i)} \Delta T_0}{R_{22}^{(i)}} \sin px \\ \tau_{xy}^{(i)} &= -p\ f_i'(y) \cos px = -p\ f_{ih}'(y) \cos px \end{aligned} \tag{13}$$

The displacements can be calculated by the integration of the constitutive relations. For the *u* displacement, from Eq. (2) :

$$\varepsilon_x^{(i)} = u_{i,x} = R_{11}^{(i)}\sigma_x + R_{12}^{(i)}\sigma_y + \alpha_x^{(i)}\Delta T \tag{14}$$

Substitution of Eqs. (8) and (11) and (12) into Eq. (14) yields :

$$u_{i,x} = \left[R_{11}^{(i)} f_{ih}''(y) - R_{12}^{(i)} p^2 f_{ih}(y) - R_{12}^{(i)} p^2 f_{ip}(y)\right]\sin px + \alpha_x^{(i)}\Delta T_0 \sin px \tag{15}$$

$$u_{i,x} = \left[R_{11}^{(i)} f_{ih}''(y) - R_{12}^{(i)} p^2 f_{ih}(y) - \left(-\alpha_x^{(i)} + \frac{\alpha_y^{(i)} R_{12}^{(i)}}{R_{22}^{(i)}}\right)\Delta T_0\right]\sin px$$

Integration in x-direction yields :

$$u_i = \left[-R_{11}^{(i)} f_{ih}''(y) + R_{12}^{(i)} p^2 f_{ih}(y) + \left(-\alpha_x^{(i)} + \frac{\alpha_y^{(i)} R_{12}^{(i)}}{R_{22}^{(i)}}\right)\Delta T_0\right]\frac{\cos px}{p} \tag{16}$$

Proceeding analogously for the v displacement :

$$\varepsilon_y^{(i)} = v_{i,y} = R_{12}^{(i)}\sigma_x + R_{22}^{(i)}\sigma_y + \alpha_y^{(i)}\Delta T \tag{17}$$

Substitution of Eqs. (8) and (11) and (12) into Eq. (17) yields :

$$v_{i,y} = \left[R_{12}^{(i)} f_{ih}''(y) - R_{22}^{(i)} p^2 f_{ih}(y) - R_{22}^{(i)} p^2 f_{ip}(y)\right]\sin px + \alpha_y^{(i)}\Delta T_0 \sin px \tag{18}$$

$$v_{i,y} = \left[R_{12}^{(i)} f_{ih}''(y) - R_{22}^{(i)} p^2 f_{ih}(y)\right]\sin px$$

Integration in y-direction yields :

$$v_i = \left[R_{12}^{(i)} f_{ih}'(y) - R_{22}^{(i)} p^2 \int f_{ih}(y)dy\right]\sin px \tag{19}$$

where $\int f_{ih}(y)dy$ is the non-definite integral of $f_i(y)$. These expressions apply to any material that respect the assumed constitutive relations in Eq. (1).

It is important to mention that in Pagano's original work [3], the p^2 coefficient in Eq. (19) is not present in the expression for the isotropic and transversely v_i displacement as it should. This point should be taken into consideretion when implementing Pagano's solution.

The homogeneous solution assumes the different shapes depending on lamina material properties. For orthotropic materials $b_i \neq 0$, then the $f_i(y)$ for the i-th layer is:

$$f_i(y) = \sum_{j=1}^{4} A_{ji}\exp(m_{ji}y_i) \;;(i = 1,2,\ldots m) \tag{20}$$

where m_{ij} coefficients are given by:

$$\left.\begin{matrix} m_{1i} \\ m_{2i} \end{matrix}\right\} = \pm p\left(\frac{a_i + b_i}{c_i}\right)^{\frac{1}{2}} \tag{21}$$

$$\left.\begin{matrix} m_{3i} \\ m_{4i} \end{matrix}\right\} = \pm p\left(\frac{a_i - b_i}{c_i}\right)^{1/2}$$

The first and second derivatives, and the indefinite integral of the $f_i(y)$, required to calculate displacements, are given by:

$$f'_{ih}(y) = \sum_{j=1}^{4} A_{ji} m_{ji} \exp(m_{ji} y_i) \; ; (i = 1,2, \dots n) \tag{22}$$

$$f'_{ih}(y) = \sum_{j=1}^{4} A_{ji} m_{ji} \exp(m_{ji} y_i) \; ; (i = 1,2, \dots n)$$

$$\int f_{ih}(y) dy = \sum_{j=1}^{4} A_{ji} \frac{1}{m_{ji}} \exp(m_{ji} y_i) \; ; (i = 1,2, \dots n)$$

If the material of a layer is isotropic or transversely isotropic in *xy* plane, b_i vanishes and the homogeneous solution becomes:

$$f_{ih}(y) = (A_{1i} + A_{2i} y_i) \exp(m_{1i} y_i) + (A_{3i} + A_{4i} y_i) \exp(-m_{1i} y_i) \tag{23}$$

with $m_{1i} = p(ai/ci)^{1/2}$. The first and second derivatives, and the indefinite integral of the $f_i(y)$, required to calculate displacements, are given by:

$$\begin{aligned} f'_{ih}(y) = {} & \big((m_{1i})A_{1i} + A_{2i}(m_{1i} y_i + 1)\big) \exp(m_{1i} y_i) \\ & + \big((-m_{1i})A_{3i} + A_{4i}(-m_{1i} y_i + 1)\big) \exp(-m_{1i} y_i) \\ f''_{ih}(y) = {} & \Big((m_{1i}^2)A_{1i} + A_{2i}\big(m_{1i}(m_{1i} y_i + 2)\big)\Big) \exp(m_{1i} y_i) \\ & + \Big((m_{1i}^2)A_{3i} + A_{4i}\big(m_{1i}(m_{1i} y_i - 2)\big)\Big) \exp(-m_{1i} y_i) \\ \int f_{ih}(y) dy = {} & \left(\left(\frac{1}{m_{1i}}\right) A_{1i} + A_{2i} \left(\frac{m_{1i} y_i - 1}{m_{1i}^2}\right)\right) \exp(m_{1i} y_i) \\ & + \left(\left(-\frac{1}{m_{1i}}\right) A_{3i} + A_{4i} \left(\frac{-m_{1i} y_i - 1}{m_{1i}^2}\right)\right) \exp(-m_{1i} y_i) \end{aligned} \tag{24}$$

In order to determine the $f_i(y)$ coefficients A_{1i}, A_{2i}, A_{3i} and A_{4i} the boundary conditions must be considered, as well as the displacement and transverse stresses continuity.

Pagano assumes simply supported edges for the 2D beam:

$$\begin{aligned} \sigma_y(0, y) = {} & \sigma_y(l, y) = 0 \\ v(0, y) = {} & v(l, y) = 0 \end{aligned} \tag{25}$$

This is automatically satisfied by Eq. (8). Therefore, there is still a total of 4*m* unknowns to be determined for a *m* layered laminate. This is achieved solving the system given by the 4*m* equations from the remnant boundary conditions and continuity equations:

$$\begin{aligned} & \sigma_y^{(1)}\left(x, \frac{h_1}{2}\right) = q_0 \sin px \; ; \; \sigma_y^{(m)}\left(x, -\frac{h_n}{2}\right) = 0 \\ & \tau_{xy}^{(1)}\left(x, \frac{h_1}{2}\right) = 0 \; ; \; \tau_{xy}^{(m)}\left(x, -\frac{h_n}{2}\right) = 0 \end{aligned} \tag{26}$$

$$\sigma_y^{(i)}\left(x,-\frac{h_i}{2}\right)=\sigma_y^{(i+1)}\left(x,\frac{h_{i+1}}{2}\right)$$
$$\tau_{xy}^{(i)}\left(x,-\frac{h_i}{2}\right)=\tau_{xy}^{(i+1)}\left(x,\frac{h_{i+1}}{2}\right)$$
$$u_i\left(x,-\frac{h_i}{2}\right)=u_{i+1}\left(x,\frac{h_{i+1}}{2}\right)$$
$$v_i\left(x,-\frac{h_i}{2}\right)=v_{i+1}\left(x,\frac{h_{i+1}}{2}\right)$$

3. NUMERICAL RESULTS

In order to assess the results of the proposed formulation, a thick 0º/90º/0º laminate under constant temperature field $\Delta T = \Delta T_0$ was evaluated. The beam dimensions are such that $l/h = 4$, where $l = 1$ m is the beam length and $h = 0.25$ m is the beam thickness. The results were compared to a commercial finite element solution using 2D plane stress four-noded bilinear elements using increasing number of elements per layer (ELPL). The 2 ELPL mesh used 120 elements (6 x 20), the 3 ELPL mesh used 288 elements (9 x 32), the 4 ELPL mesh used 480 elements (12 x 40), the 5 ELPL mesh used 780 elements (15 x 52) and the 30 ELPL mesh used 32400 elements (90 x 360).

In order to represent the constant temperature distribution, the Fourier expansion required is:

$$\Delta T\ (x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Delta T_0 \sin px \qquad (27)$$

with $\Delta T_0 = 1\ K$.

As the representation is more accurate as more terms are considered, 3001 terms were considered. Also, as the sinusoidal expansion is problematic on the beam edges, the results were evaluated at $x = 0.25l$, where the boundary effects are not representative.

The mechanical properties of the unidirectional lamina are [7]:

$E_L = 150.0$ GPa; $E_T = 10.0$ GPa;

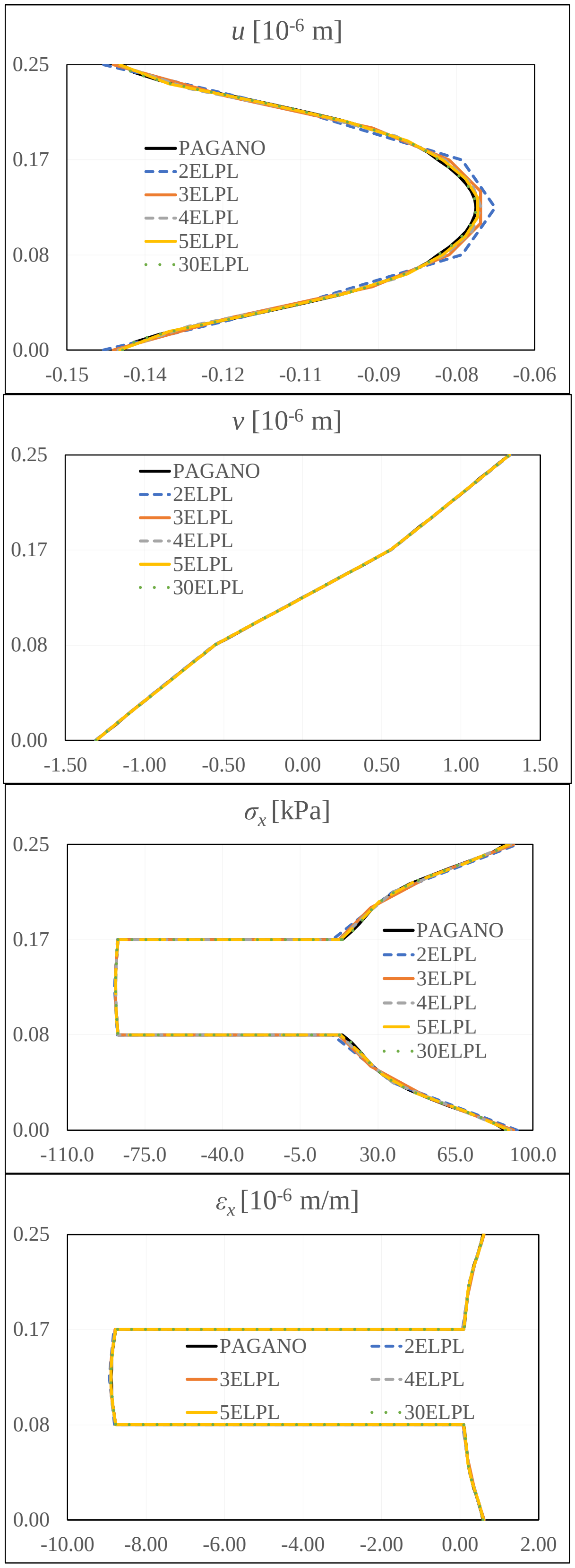
$\nu_{LT} = 0.3$; $\nu_{TT} = 0.48$;

$G_{LT} = 5.0$ GPa; $G_{TT} = 3.378$ GPa;

$\alpha_L = 0.139\ 10^{-6}\ K^{-1}$; $\alpha_T = 9.0\ 10^{-6}\ K^{-1}$;

where the superscript l denotes the longitudinal direction (fibre direction) and the t denotes the transverse direction.

The finite element convergence is attested as 30ELPL mesh meets perfectly the Pagano's solution.



u [10-6 m]
PAGANO
2ELPL
3ELPL
4ELPL
5ELPL
30ELPL
0.25
0.17
0.08
0.00
-0.15
-0.14
-0.12
-0.11
-0.09
-0.08
-0.06
v [10-6 m]
-1.50
-1.00
-0.50
0.00
0.50
1.00
1.50
σx [kPa]
-110.0
-75.0
-40.0
-5.0
30.0
65.0
100.0
εx [10-6 m/m]
-10.00
-8.00
-6.00
-4.00
-2.00
0.00
2.00

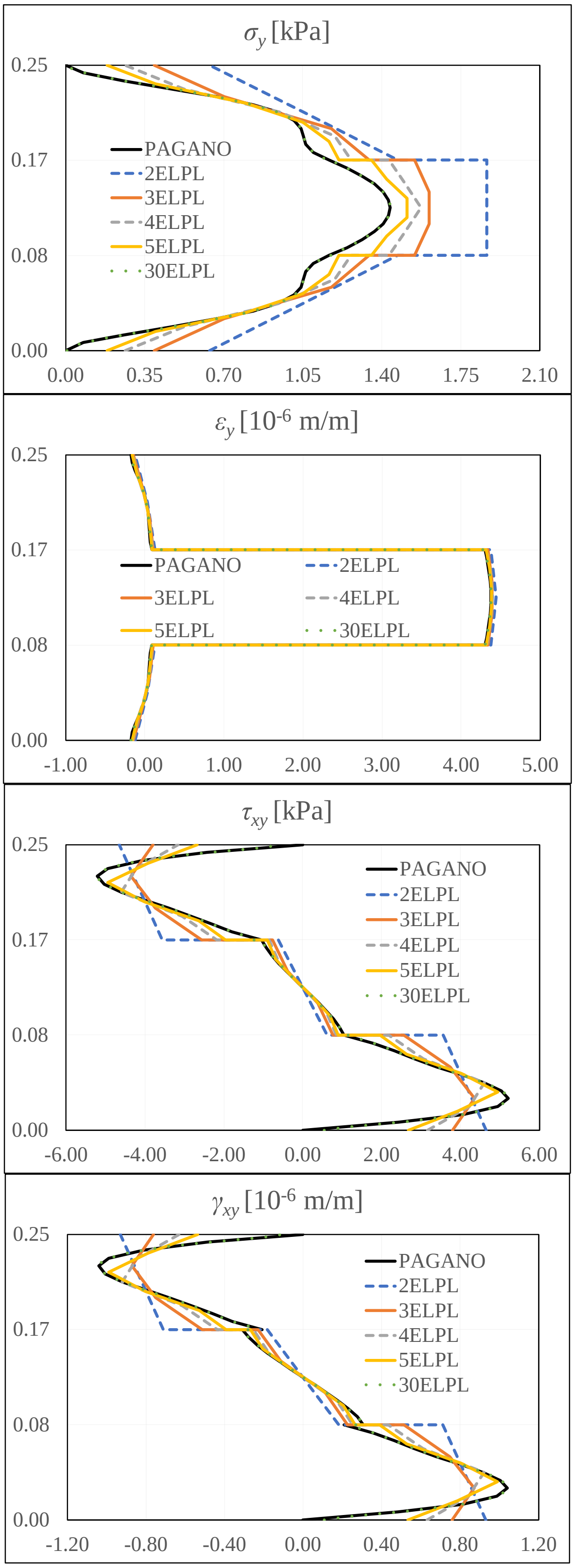


Figure 1: Analytical solution and finite element results.

It is common in composite problems to model structures using one element per layer [8], but by the results illustrated in Fig. 1 it is clear that, even for a simple problem such as the one evaluated, using few elements per layer is a poor approximation, especially considering transverse strains and stresses. One can see that even using five elements per layer the value of σ_y stress, critical to delamination problems, is overestimated.

4. CONCLUSIONS

The present work is a straightforward development of Pagano's solution for 2D solution of laminated composite beam under cylindrical bending including thermal-effects and is a reliable reference for the development of composite structure theories and models.

An important conclusion from the primary application of the model in contrast to traditional finite element modelling is that, in order to have accurate results concerning transverse strains and stresses, more than five elements per layer are required. Therefore, common strategies for modelling composite structures must be carefully studied.

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